

COVERING RESULTS AND PERTURBED ROPER–SUFFRIDGE OPERATORS

MARK ELIN AND MARINA LEVENSHTAIN

ABSTRACT. This work is devoted to the advanced study of Roper–Suffridge type extension operators. For a given non-normalized spirallike function (with respect to an interior or boundary point) on the open unit disk of the complex plane, we construct perturbed extension operators in a certain class of Banach spaces and prove that these operators preserve the spirallikeness property. In addition, we present an extension operator for semigroup generators.

We use a new geometric approach based on the connection between spirallike mappings and one-parameter continuous semigroups. It turns out that the new one-dimensional covering results established below are crucial for our investigation.

1. INTRODUCTION

In the last fifteen years, many authors have devoted their works to the study of extension operators in the following sense. Given a family of biholomorphic mappings on the unit ball of a proper subspace of a Banach space X , to extend it to a family of biholomorphic mappings of the unit ball of the whole space X with preserving some required geometric properties such as convexity, starlikeness, spirallikeness and others. These investigations were initiated in 1995 with the paper by Roper and Suffridge [14], where they introduced an extension operator which preserves a variety of required geometric characteristics. Namely, given a (locally) univalent function h on the open unit disk $\Delta := \{z : |z| < 1\}$ of the complex plane \mathbb{C} normalized by $h(0) = h'(0) - 1 = 0$, they considered the mapping $\Phi[h] : \mathbb{B}^n \mapsto \mathbb{C}^n$ on the open unit ball \mathbb{B}^n in \mathbb{C}^n defined as follows:

$$h \mapsto \Phi[h](x, y) = \left(h(x), \sqrt{h'(x)} y \right), \quad (1)$$

where $y \in \mathbb{C}^{n-1}$, $|x|^2 + \|y\|^2 < 1$. This operator preserves convexity [14], starlikeness [7], μ -spirallikeness [9, 10], as well as other important properties.

Key words and phrases. extension operator, continuous semigroup, infinitesimal generator, spirallike mapping, starlike mapping.

Later, this idea was developed in different directions (see, for example, [8, 5, 2] and references therein), diverse generalizations and modifications of this operator were considered in various normed spaces.

Recently, the first author [2] introduced a general construction which involves most extension operators considered earlier, namely, the operator

$$h \mapsto \Phi_\Gamma[h](x, y) := (h(x), \Gamma(h, x)y) \quad (2)$$

defined on the unit ball of the direct product $X \times Y$ of two Banach spaces, where $\Gamma(h, x)$ is an operator-valued mapping satisfying some natural conditions. It was proved in [2] that these conditions provide that the extension operator $\Phi_\Gamma[h]$ preserves starlikeness and spirallikeness.

In [11, 12], Muir considered an extension operator which is not included in construction (2). More precisely, given a univalent function h on Δ normalized by $h(0) = h'(0) - 1 = 0$, he introduced the operator $\widehat{\Phi}[h] : \mathbb{B}^n \mapsto \mathbb{C}^n$ defined by

$$h \mapsto \widehat{\Phi}[h](x, y) = \left(h(x) + h'(x)Q(y), \sqrt{h'(x)}y \right), \quad (3)$$

where Q is a homogeneous quadratic polynomial and proved, in particular, that $\widehat{\Phi}$ preserves starlikeness whenever $\sup_{\|y\|=1} |Q(y)| \leq \frac{1}{4}$. The proof of the last result is carried out in [11] by a technical verification of Suffridge's starlikeness criterion (see [15]).

In Section 3, we give a geometric explanation of this fact. By the way, this answers question (c) in [2]. It turns out that the mentioned above Muir's theorem follows immediately from some new one-dimensional covering theorems established in Section 4. Section 5 is devoted to an advanced study of the Roper–Suffridge type extension operators. Moreover, the semigroup approach enables us to study special perturbed extension operators in a certain class of Banach spaces in Section 6. These operators preserve spirallikeness (with respect to an arbitrary interior or boundary point).

2. PRELIMINARY NOTIONS

In this section we present some notions of nonlinear analysis and geometric function theory which will subsequently be useful.

Let X be a complex Banach space with the norm $\|\cdot\|$. Denote by $\text{Hol}(D, E)$ the set of all holomorphic mappings on a domain $D \subseteq X$ which map D into a set $E \subseteq X$, and set $\text{Hol}(D) := \text{Hol}(D, D)$.

We start with the notion of a one-parameter continuous semigroup.

Definition 1. Let D be a domain in a complex Banach space X . A family $S = \{F_t\}_{t \geq 0} \subset \text{Hol}(D)$ of holomorphic self-mappings of D is said to be a one-parameter continuous semigroup (in short, semigroup) on D if

$$F_{t+s} = F_t \circ F_s, \quad t, s \geq 0, \quad (4)$$

and for all $x \in D$,

$$\lim_{t \rightarrow 0^+} F_t(x) = x. \quad (5)$$

Definition 2. A semigroup $S = \{F_t\}_{t \geq 0}$ on D is said to be generated if for each $x \in D$, there exists the strong limit

$$f(x) := \lim_{t \rightarrow 0^+} \frac{1}{t} \left(x - F_t(x) \right). \quad (6)$$

In this case the mapping $f : D \mapsto X$ is called the (infinitesimal) generator of the semigroup S .

In fact, semigroups are one of the key tools in the geometric function theory. For instance, a mapping h is spirallike if there exists a linear semigroup on its image. More precisely:

Definition 3. Let h be a biholomorphic mapping defined on a domain D of a Banach space X . The mapping h is said to be spirallike if there is a bounded linear operator A such that the function $\text{Re } \lambda$ is bounded away from zero on the spectrum of A and such that for each point $w \in h(D)$ and each $t \geq 0$, the point $e^{-tA}w$ also belongs to $h(D)$. In this case h is called A -spirallike. If A can be chosen to be the identity mapping, that is, $e^{-t}w \in h(D)$ for all $w \in h(D)$ and all $t \geq 0$, then h is called starlike.

It follows by this definition that the origin belongs to the closure $\overline{h(D)}$. In the case where the image $h(D)$ contains the origin, we say that h is spirallike with respect to an interior point. Otherwise, if the origin belongs to the boundary of $h(D)$, h is said to be spirallike with respect to a boundary point.

In the one-dimensional case, a univalent function $h : \Delta \mapsto \mathbb{C}$ is μ -spirallike ($\text{Re } \mu > 0$) if for all $t \geq 0$,

$$e^{-\mu t} h(\Delta) \subseteq h(\Delta).$$

If this inclusion holds for $\mu = 1$, the function h is starlike.

3. GEOMETRIC INTERPRETATION

Let h be a normalized starlike function on the open unit disk of the complex plane. By Theorem 4.1 in [11], the mapping $\widehat{\Phi}[h]$ defined by (3) is the starlike mapping of the open unit ball in \mathbb{C}^n whenever $\sup_{\|y\|=1} |Q(y)| \leq \frac{1}{4}$. This means that its image is invariant under the action of the semigroup $\{G_t\}_{t \geq 0}$, where

$$G_t(z, w) = (e^{-t}z, e^{-t}w), \quad z \in \mathbb{C}, \quad w \in \mathbb{C}^{n-1}.$$

In addition, it is easy to see that $\widehat{\Phi}[h] = \Psi \circ \Phi[h]$, where $\Phi[h]$ is the Roper–Suffridge extension operator defined by (1) and $\Psi(z, w) = (z + Q(w), w)$ is the automorphism of the space $\mathbb{C} \times \mathbb{C}^{n-1}$. Therefore, the family $\{F_t\}_{t \geq 0}$ defined by $F_t := \Psi^{-1} \circ G_t \circ \Psi$ forms a semigroup on the image of the mapping $\Phi[h]$. A direct calculation shows that

$$F_t(z, w) = (e^{-t}z + (e^{-t} - e^{-2t})Q(w), e^{-t}w). \quad (7)$$

On the other hand, it is known that the mapping $\Phi[h]$ is starlike [7], i.e., its image is invariant under the action of the semigroup

$$(z, w) \mapsto (e^{-t}z, e^{-t}w). \quad (8)$$

Comparing semigroup elements (7) and (8), we see that both of them have the same second coordinate.

Fix a point (z_0, w_0) in the image of $\Phi[h]$ and $t > 0$. Consider the preimage of the intersection of $\Phi[h](\mathbb{B}^n)$ with $w = e^{-t}w_0$. This preimage is a submanifold of the ball \mathbb{B}^n . Denote by Ω the orthogonal projection of this submanifold on the first coordinate plane. Taking into account that the first coordinate of the Roper–Suffridge extension operator is $h(x)$, we conclude that the mentioned Muir’s result is equivalent to the fact that $h(\Omega)$ covers some disk centered at $e^{-t}z_0$.

According to this explanation, we start our consideration from two auxiliary one-dimensional covering results which are of independent interest.

4. COVERING RESULTS

In this section, we consider non-normalized univalent functions $h \in \text{Hol}(\Delta, \mathbb{C})$ on the open unit disk Δ . For some special subsets Ω_α of Δ , we determine disks covered by $h(\Omega_\alpha)$.

Theorem 1. *Let $h \in \text{Univ}(\Delta, \mathbb{C})$. For $\alpha \in (0, 1)$, and $x_0 \in \Delta$ define the set*

$$\Omega_\alpha := \{x \in \Delta : \alpha|h'(x_0)|(1 - |x_0|^2) < |h'(x)|(1 - |x|^2)\}.$$

Then the image $h(\Omega_\alpha)$ covers the open disk of radius $\frac{1-\alpha}{4}|h'(x_0)|(1-|x_0|^2)$ centered at $h(x_0)$.

Proof. Consider the function $g : \Delta \mapsto \mathbb{C}$ defined by

$$g(z) := \frac{h\left(\frac{x_0 - z}{1 - \overline{x_0}z}\right) - h(x_0)}{h'(x_0)(|x_0|^2 - 1)}, \quad z \in \Delta.$$

Since g is a normalized univalent function, by the Koebe One-Quarter Theorem, $g(\Delta)$ covers the disk $|z| < \frac{1}{4}$ and, consequently, $h(\Delta)$ covers the open disk of radius $\frac{1}{4}|h'(x_0)|(1-|x_0|^2)$ centered at $h(x_0)$.

Standard calculations show that $x := \frac{x_0 - z}{1 - \overline{x_0}z} \in \Omega_\alpha$ if and only if

$$z \in \tilde{\Omega}_\alpha := \left\{ z \in \Delta : |g'(z)| > \frac{\alpha}{1 - |z|^2} \right\}. \quad (9)$$

On the other hand, by the Koebe Distortion Theorem, for each $z \in \Delta$,

$$|g'(z)| \geq \frac{1 - |z|}{(1 + |z|)^3} \quad (10)$$

and

$$|g(z)| \geq \frac{|z|}{(1 + |z|)^2}. \quad (11)$$

It follows from (9) and (10), that for each $z \in \Delta \setminus \tilde{\Omega}_\alpha$,

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |g'(z)| \leq \frac{\alpha}{1 - |z|^2},$$

hence,

$$(1 - \alpha)|z|^2 - 2(1 + \alpha)|z| + (1 - \alpha) \leq 0,$$

which implies $|z| \geq \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}}$.

Therefore, for each point $z \in \Delta \setminus \tilde{\Omega}_\alpha$, by inequality (11),

$$|g(z)| \geq \frac{|z|}{(1 + |z|)^2} \geq \frac{\frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}}}{\left(1 + \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}}\right)^2} = \frac{1 - \alpha}{4}.$$

Let now $x^* \in \Delta \setminus \Omega_\alpha$. Then $z^* := \frac{x_0 - x^*}{1 - \overline{x_0}x^*} \in \Delta \setminus \tilde{\Omega}_\alpha$ and hence

$$|g(z^*)| = \frac{|h(x^*) - h(x_0)|}{|h'(x_0)|(1 - |x_0|^2)} \geq \frac{1 - \alpha}{4},$$

or

$$|h(x^*) - h(x_0)| \geq \frac{1-\alpha}{4} |h'(x_0)|(1-|x_0|^2),$$

and the assertion follows. ■

Theorem 2. *Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ be such that $0 < \alpha < |\beta| < 1$, and suppose that $h \in \text{Univ}(\Delta, \mathbb{C})$ satisfies $\beta h(\Delta) \subseteq h(\Delta)$. Given a point $x_0 \in \Delta$, define the point $x_1 = h^{-1}(\beta h(x_0)) \in \Delta$ and the set*

$$\Omega_\alpha := \{x \in \Delta : \alpha |h'(x_0)|(1-|x_0|^2) < |h'(x)|(1-|x|^2)\}.$$

Then the image $h(\Omega_\alpha)$ covers the disk centered at $h(x_1) = \beta h(x_0)$ of radius $\frac{|\beta|-\alpha}{4|\beta|} |h'(x_1)|(1-|x_1|^2)$ which is not less than $R = \frac{|\beta|-\alpha}{4} |h'(x_0)|(1-|x_0|^2)$.

Proof. By our assumptions, the function $F := h^{-1}(\beta h)$ is a holomorphic self-mapping of Δ and, consequently, satisfies the Schwarz–Pick inequality:

$$|F'(x_0)| \leq \frac{1-|F(x_0)|^2}{1-|x_0|^2} = \frac{1-|x_1|^2}{1-|x_0|^2}.$$

Set $g := h \circ F = \beta h$. Then

$$|g'(x_0)|(1-|x_0|^2) = |h'(x_1)| \cdot |F'(x_0)|(1-|x_0|^2) \leq |h'(x_1)|(1-|x_1|^2). \quad (12)$$

On the other hand, $|g'(x_0)| = |\beta| \cdot |h'(x_0)|$ and so

$$|\beta| \cdot |h'(x_0)|(1-|x_0|^2) \leq |h'(x_1)|(1-|x_1|^2).$$

Hence, the set

$$\Omega := \left\{x \in \Delta : \frac{\alpha}{|\beta|} \cdot |h'(x_1)|(1-|x_1|^2) < |h'(x)|(1-|x|^2)\right\}$$

is a subset of Ω_α . By Theorem 1, the disk centered at $h(x_1)$ of radius $\frac{1-\frac{\alpha}{|\beta|}}{4} |h'(x_1)|(1-|x_1|^2)$ lies in $h(\Omega)$, and the assumption follows. ■

Remark 1. Note that the condition $\beta h(\Delta) \subseteq h(\Delta)$ with $|\beta| < 1$ is necessary and sufficient for a holomorphic mapping $h : \Delta \mapsto \mathbb{C}$ to be a solution to the so-called Schröder functional equation

$$h \circ F = \beta h, \quad (13)$$

where F is a holomorphic self-mapping of the open unit disk Δ of dilation or hyperbolic type. In the dilation case, the Denjoy–Wolff point τ of F is an interior point of Δ . It is clear that if (13) has a solution, then $\beta = F'(\tau)$. It was proved by Koenigs that the last equality is also

sufficient for the solvability of the Schröder equation (13) whenever $|\beta| \neq 0, 1$. Otherwise, when F is of hyperbolic type, the Denjoy–Wolff point τ belongs to the boundary $\partial\Delta$ and $F'(\tau) := \lim_{r \rightarrow 1^-} \frac{F(r\tau) - \tau}{\tau(r-1)}$ exists and is a positive number less than 1. For the case $\beta = F'(\tau)$ the existence of a solution to equation (13) was proved by Valiron. The survey of the related topics can be found in [4].

On the other hand, if the function h is μ -spirallike then the inclusion $\beta h(\Delta) \subseteq h(\Delta)$ holds for each β of the form $\beta = e^{-t\mu}$, $t > 0$. ►

To finish this section, we remark that according to the geometric explanation given in Section 3, Muir’s theorem on starlikeness preserving follows from Theorem 2 with $\alpha = e^{-2t}$ and $\beta = e^{-t}$. In the next section, we consider a more general situation.

5. ROPER-SUFFRIDGE TYPE EXTENSIONS

Let $(Y, \|\cdot\|_Y)$ be a complex Banach space and let $r \geq 1$. Consider the set

$$\mathbb{B} := \{(x, y) \in \mathbb{C} \times Y : |x|^2 + \|y\|_Y^r < 1\}.$$

This set is the open unit ball of $\mathbb{C} \times Y$ with respect to a norm $\|\cdot\|$. Actually, this norm is the Minkowski functional of the set \mathbb{B} . Obviously, $\mathbb{C} \times Y$ equipped with this norm $\|\cdot\|$ is a complex Banach space.

It follows from Theorem 5.1 in [2] that the extension $H(x, y) = (h(x), h'(x)^{\frac{1}{r}}y)$ of a μ -spirallike function h on Δ is $\begin{pmatrix} \mu & 0 \\ 0 & B + \frac{\mu}{r}\text{id}_Y \end{pmatrix}$ -spirallike for each linear accretive operator B on Y (note in passing that the branch of the power $h'(x)^{\frac{1}{r}}$ can be chosen appropriately). We apply covering results from the previous section to the advanced study of this operator. Our main result (Theorem 3) provides a new tool for the study of various extension operators. In particular, in the next section we apply it to prove that Muir’s operator preserves spirallikeness in a certain class of Banach spaces.

Theorem 3. *Let $h : \Delta \mapsto \mathbb{C}$ be a μ -spirallike mapping and $H : \mathbb{B} \mapsto \mathbb{C} \times Y$ be defined by*

$$H(x, y) := \left(h(x), h'(x)^{\frac{1}{r}}y \right). \quad (14)$$

Suppose that $B \in L(Y)$ generates a semigroup of strict contractions. Then for each point $(z_0, w_0) \in H(\mathbb{B})$ and for all $t > 0$,

$$\left(e^{-\mu t} z_0 + \gamma_t, e^{-\frac{\mu}{r} t} e^{-Bt} w_0 \right) \in H(\mathbb{B})$$

whenever $|\gamma_t| < R_t = \frac{1 - \|e^{-Bt}\|^r}{4} |h'(x_1)| (1 - |x_1|^2)$ with $x_1 = h^{-1}(e^{-\mu t} z_0)$.

Note that if the operator B is scalar, i.e., $B = \lambda \text{id}_Y$ ($\lambda \in \mathbb{C}$), it generates a semigroup of strict contractions if and only if $\text{Re } \lambda > 0$.

Proof. Since H is a $\begin{pmatrix} \mu & 0 \\ 0 & B + \frac{\mu}{r} \text{id}_Y \end{pmatrix}$ -spirallike mapping (see [2]), its image $H(\mathbb{B})$ is invariant under the action of the semigroup $\{F_t\}_{t \geq 0}$ defined by

$$F_t(z, w) = \left(e^{-\mu t} z, e^{-\frac{\mu}{r} t} e^{-Bt} w \right), \quad (z, w) \in H(\mathbb{B}), \quad t \geq 0. \quad (15)$$

Fix $t > 0$ and $(z_0, w_0) \in H(\mathbb{B})$. Then $z_0 = h(x_0)$ and $w_0 = h'(x_0)^{\frac{1}{r}} y_0$ for some $(x_0, y_0) \in \mathbb{B}$.

Denote $(z_1, w_1) := (e^{-\mu t} z_0, e^{-\frac{\mu}{r} t} e^{-Bt} w_0) \in H(\mathbb{B})$. Then there is a point $(x_1, y_1) \in \mathbb{B}$ such that $z_1 = h(x_1)$ and $w_1 = h'(x_1)^{\frac{1}{r}} y_1$. Let U be a region in the complex plane \mathbb{C} containing z_1 and such that for all $z \in U$, the point (z, w_1) belongs to the image $H(\mathbb{B})$. We have to estimate disks contained in U .

Consider the set of all $(x, y) \in \mathbb{B}$ such that $h'(x)^{\frac{1}{r}} y = w_1$ and, consequently, $h(x) \in U$. Since $|x|^2 + \|y\|_Y^r < 1$, we have $\|w_1\|^r < |h'(x)|(1 - |x|^2)$.

On the other hand, $w_1 = e^{-\frac{\mu}{r} t} e^{-Bt} w_0 = e^{-\frac{\mu}{r} t} e^{-Bt} h'(x_0)^{\frac{1}{r}} y_0$. Therefore,

$$|e^{-\mu t}| \cdot \|e^{-Bt}\|^r \cdot |h'(x_0)| \cdot \|y_0\|_Y^r < |h'(x)|(1 - |x|^2). \quad (16)$$

Define the set

$$\Omega := \{x \in \Delta : |e^{-\mu t}| \cdot \|e^{-Bt}\|^r \cdot |h'(x_0)| \cdot (1 - |x_0|^2) < |h'(x)|(1 - |x|^2)\}.$$

It is clear that for each $x \in \Omega$ and $y = \frac{1}{h'(x)^{\frac{1}{r}}} w_1$, we have $(x, y) \in \mathbb{B}$ and $h'(x)^{\frac{1}{r}} y = w_1$. Thus, $h(\Omega) \subseteq U$.

By Theorem 2 with $\beta = e^{-\mu t}$ and $\alpha = |e^{-\mu t}| \cdot \|e^{-Bt}\|^r$, the image $h(\Omega)$ covers the open disk centered at $z_1 = e^{-\mu t} z_0$ of radius $\frac{1 - \|e^{-Bt}\|^r}{4} |h'(x_1)| \cdot (1 - |x_1|^2)$. The proof is complete. ■

Corollary 1. *Let $h : \Delta \mapsto \mathbb{C}$ be μ -spirallike and A be a linear operator in $\mathbb{C} \times Y$ with $\text{Re } \sigma > \varepsilon > 0$ for each σ from the spectrum of A . Suppose that there exists an automorphism Φ of the space $\mathbb{C} \times Y$ such that for each point $(z_0, w_0) \in H(\mathbb{B})$ and for all $t \geq 0$,*

$$\Phi^{-1} (e^{-tA} \Phi(z_0, w_0)) = \left(z_t, e^{-\frac{\mu}{r} t} e^{-Bt} w_0 \right), \quad (17)$$

for some $B \in L(Y)$ which generates a semigroup of strict contractions. If

$$|z_t - e^{-\mu t} z_0| \leq R_t = \frac{1 - \|e^{-Bt}\|^r}{4} |h'(x_1)| (1 - |x_1|^2), \quad (18)$$

where $x_1 = h^{-1}(e^{-\mu t} z_0)$, then the mapping $\Phi \circ H : \mathbb{B} \mapsto \mathbb{C} \times Y$ is A -spirallike.

Proof. Denote $\gamma_t := z_t - e^{-\mu t} z_0$ and rewrite equality (17) in the form

$$\Phi^{-1}(e^{-tA}\Phi(z_0, w_0)) = \left(e^{-\mu t} z_0 + \gamma_t, e^{-\frac{\mu}{r}t} e^{-Bt} w_0 \right).$$

Since $|\gamma_t| \leq R_t$ by (18), it follows from Theorem 3 that for all $t \geq 0$,

$$\Phi^{-1}(e^{-tA}\Phi(z_0, w_0)) \subseteq H(\mathbb{B})$$

or, which is the same, $e^{-tA}\Phi(z_0, w_0) \subseteq \Phi(H(\mathbb{B}))$. ■

Remark 2. It follows from the proof of Theorem 2 that R_t defined in Theorem 3 and Corollary 1 is not less than $|e^{-\mu t}| \frac{1 - \|e^{-Bt}\|^r}{4} |h'(x_0)| \cdot (1 - |x_0|^2)$. ►

6. MUIR'S TYPE PERTURBATIONS OF EXTENSION OPERATORS

In this section, we present some generalizations of Muir's perturbations of the Roper–Suffridge operators preserving spirallikeness. It is worth mentioning that in saying ‘spirallike’, we mean spirallikeness with respect to an arbitrary interior or boundary point and do not separate these two different cases. Further, we construct extensions for holomorphic generators on Δ of hyperbolic and dilation type which, in their turn, generate semigroups on the unit balls of the corresponding Banach spaces.

As above, $(Y, \|\cdot\|_Y)$ is a complex Banach space, and $\mathbb{C} \times Y$ is the Banach space with the open unit ball $\mathbb{B} = \{(x, y) \in \mathbb{C} \times Y : |x|^2 + \|y\|^r < 1\}$ for some $r \geq 1$. For simplicity, we will use the results of the previous section in the particular case when B is a scalar operator $B = \lambda \text{id}_Y$.

Theorem 4. Let $h : \Delta \mapsto \mathbb{C}$ be μ -spirallike and $Q : Y \mapsto \mathbb{C}$ be a homogeneous polynomial of degree $r \in \mathbb{N}$. Then the mapping $\hat{H} : \mathbb{B} \mapsto \mathbb{C} \times Y$ defined by

$$\hat{H}(x, y) := \left(h(x) + h'(x)Q(y), h'(x)^{\frac{1}{r}} y \right)$$

is $\begin{pmatrix} \mu & 0 \\ 0 & (\lambda + \frac{\mu}{r})\text{id}_Y \end{pmatrix}$ -spirallike for each $\lambda \in \mathbb{C}$, $\text{Re } \lambda > 0$, whenever

$$\sup_{\|y\|=1} |Q(y)| \leq \frac{1}{4} \cdot \frac{\text{Re } \lambda}{|\lambda|}.$$

Moreover, this bound is sharp.

Proof. The mapping \widehat{H} can be represented in the form

$$\widehat{H} = \Phi \circ H,$$

where $\Phi(z, w) := (z + Q(w), w)$ is the automorphism of the space $\mathbb{C} \times Y$ and H is defined by (14).

We have to show that for each element F_t of the semigroup $\{F_t\}_{t \geq 0}$ defined by (15), the inclusion

$$F_t \circ \Phi(H(\mathbb{B})) \subseteq \Phi(H(\mathbb{B}))$$

holds or, which is the same, for each point $(z_0, w_0) \in H(\mathbb{B})$ and for all $t > 0$,

$$\begin{aligned} & \Phi^{-1}(F_t \circ \Phi(z_0, w_0)) \\ &= \left(e^{-\mu t} z_0 + (e^{-\mu t} - e^{-(\lambda + \frac{\mu}{r})rt}) Q(w_0), e^{-(\lambda + \frac{\mu}{r})t} w_0 \right) \in H(\mathbb{B}). \end{aligned}$$

Fix $t > 0$ and $(z_0, w_0) \in H(\mathbb{B})$, and denote $\gamma_t := (e^{-\mu t} - e^{-(\lambda + \frac{\mu}{r})rt}) Q(w_0)$. By our assumption,

$$|Q(w_0)| < \frac{1}{4} \frac{\text{Re } \lambda}{|\lambda|} |h'(x_0)| (1 - |x_0|^2),$$

where $x_0 = h^{-1}(z_0)$. Therefore, by Corollary 1 and Remark 2, it suffices to show that

$$|1 - e^{-r\lambda t}| \frac{\text{Re } \lambda}{|\lambda|} < 1 - |e^{-r\lambda t}|. \quad (19)$$

To this end, we denote

$$f(t) := \left(\frac{1 - |e^{-r\lambda t}|}{|1 - e^{-r\lambda t}|} \right)^2$$

and prove that $\inf_{t>0} f(t) = \left(\frac{\text{Re } \lambda}{|\lambda|} \right)^2$.

Indeed, $\lim_{t \rightarrow \infty} f(t) = 1$ and $\lim_{t \rightarrow 0} f(t) = \left(\frac{\text{Re } \lambda}{|\lambda|} \right)^2$.

Moreover, if we denote $a := \text{Re } \lambda$ and $b := \text{Im } \lambda$, then the derivative $f'(t)$ vanishes either when $\cos(brt) = 1$ and then $f(t) = 1$, or when

$$\cos(brt) = \frac{a^2(1 + e^{-art})^2 - b^2(1 - e^{-art})^2}{a^2(1 + e^{-art})^2 + b^2(1 - e^{-art})^2} \text{ and then}$$

$$f(t) = \frac{a^2}{a^2 + b^2} + \frac{b^2(1 - e^{-art})^2}{a^2(1 + e^{-art})^2 + b^2(1 - e^{-art})^2} > \frac{a^2}{a^2 + b^2} = \left(\frac{\operatorname{Re} \lambda}{|\lambda|} \right)^2.$$

The proof is complete. ■

Now we consider extension operators for semigroup generators.

Theorem 5. *Let $f : \Delta \mapsto \mathbb{C}$ be the generator of a semigroup on $\overline{\Delta}$ of dilation or hyperbolic type having the Denjoy–Wolff point $\tau \in \overline{\Delta}$ and $Q : Y \mapsto \mathbb{C}$ be a homogeneous polynomial of degree $r \in \mathbb{N}$. Denote $\mu := f'(\tau)$. Then for each $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda > 0$, the mapping $\widehat{f} : \mathbb{B} \mapsto \mathbb{C} \times Y$ defined by*

$$\widehat{f}(x, y) := \left(f(x) + Q(y), \frac{1}{r} \left(f'(x) + r\lambda - \frac{\mu - f'(x)}{f(x)} Q(y) \right) y \right) \quad (20)$$

is a semigroup generator on \mathbb{B} whenever $\sup_{\|y\|=1} |Q(y)| \leq \frac{r \operatorname{Re} \lambda}{4}$.

Proof. Define the function $h : \Delta \mapsto \mathbb{C}$ as a solution of the differential equation (see [4])

$$h'(z)f(z) = \mu h(z), \quad z \in \Delta. \quad (21)$$

It is known that h is a univalent function. Moreover, if f is the generator of a dilation type semigroup, the function h is μ -spirallike with respect to an interior point. Otherwise, if the generated semigroup is of hyperbolic type, h is μ -spirallike with respect to a boundary point.

By Corollary 4, the mapping $\widetilde{H} : \mathbb{B} \mapsto \mathbb{C} \times Y$ defined by

$$\widetilde{H}(x, y) := \left(h(x) - \frac{h'(x)}{r\lambda} Q(y), h'(x)^{\frac{1}{r}} y \right)$$

is $\begin{pmatrix} \mu & 0 \\ 0 & (\lambda + \frac{\mu}{r}) \operatorname{id}_Y \end{pmatrix}$ -spirallike. Consequently, the family $\{\widetilde{F}_t\}_{t \geq 0}$ of holomorphic mappings

$$\widetilde{F}_t(z, w) := \left(e^{-\mu t} z, e^{-t(\lambda + \frac{\mu}{r}) \operatorname{id}_Y} w \right),$$

is a semigroup on $\widetilde{H}(\mathbb{B})$. Differentiating $\widetilde{F}_t(z, w)$ at $t = 0^+$, we find its generator

$$\widetilde{f}(z, w) = -\frac{d}{dt} \widetilde{F}_t(z, w)|_{t=0^+} = \left(\mu z, \left(\lambda + \frac{\mu}{r} \right) w \right).$$

By Lemma 3.7.1 on p. 30 of [3], the mapping $\widehat{f}: \mathbb{B} \mapsto \mathbb{C} \times Y$ defined by

$$\widehat{f}(x, y) := [D\widetilde{H}(x, y)]^{-1} \widetilde{f}(\widetilde{H}(x, y)),$$

is a semigroup generator on \mathbb{B} .

By straightforward calculations we get

$$[D\widetilde{H}(x, y)]^{-1} = \begin{pmatrix} \frac{1}{h'(x)} & \frac{1}{r\lambda h'(x)^{\frac{1}{r}}} Q'(y) \\ \frac{h''(x)}{-r h'(x)^2} & \frac{r\lambda h'(x) - h''(x)Q(y)}{r\lambda h'(x)^{1+\frac{1}{r}}} id_Y \end{pmatrix},$$

and thus

$$\begin{aligned} \widehat{f}(x, y) &= \begin{pmatrix} \frac{1}{h'(x)} & \frac{1}{r\lambda h'(x)^{\frac{1}{r}}} Q'(y) \\ \frac{h''(x)}{-r h'(x)^2} & \frac{r\lambda h'(x) - h''(x)Q(y)}{r\lambda h'(x)^{1+\frac{1}{r}}} id_Y \end{pmatrix} \widetilde{f}(\widetilde{H}(x, y)) \\ &= \left(f(x) + Q(y), \frac{1}{r} (f'(x) + r\lambda - \frac{\mu - f'(x)}{f(x)} Q(y)) y \right), \end{aligned} \quad (22)$$

which complete the proof. ■

Remark 3. Note that the condition $\mu = f'(\tau)$ was used in the proof of Theorem 5 for the solvability of equation (21) only. Since in the hyperbolic case this equation is solvable for each μ such that $|\mu - f'(\tau)| \leq f'(\tau)$, $\mu \neq 0$ (see [4]), we conclude that in such a situation, our assertion also holds. ►

Remark 4. In the last two theorems, we required for r to be integer by the only reason in order to define homogeneous polynomial Q of degree r . At the same time, the polynomial $Q \equiv 0$ obviously satisfies all the conditions of the theorems. So, the conclusions of Theorems 4 and 5 hold with $Q \equiv 0$ and arbitrary $r \geq 1$. ►

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DEPARTMENT OF MATHEMATICS, ORT BRAUDE COLLEGE, KARMIEL 21982,
ISRAEL

E-mail address: mark-elin@braude.ac.il

DEPARTMENT OF MATHEMATICS, ORT BRAUDE COLLEGE, KARMIEL 21982,
ISRAEL

E-mail address: marlev@braude.ac.il